

Anti-de Sitter-type 2+1 spacetime of a charged rotating mass

Nikolai V. Mitskievich* and Alberto A. García†

Abstract

The exact charged rotating solution of 2+1 Einstein–Maxwell equations with Λ term is obtained and its properties outlined. It generalizes the Cataldo–Cruz–del Campo–García relativistic charged massive black hole on the 2+1 anti-de Sitter cosmological background. We show that rotating solutions correspond to inhomogeneous field equations, thus presence of sources in 2+1 Maxwell’s equations cannot be identified with existence of a charge distribution. Instead, these sources are related to the 2+1 Machian 2-form field, and the overall Lagrangian structure of the rotating system is reconstructed.

In 2+1 spacetime the 1-form field has to obey nonlinear equations to yield zero-trace stress-energy tensor (we call this the intrinsically relativistic field condition). This automatically guarantees a profound analogy of its spherically symmetric static electrovacuum solution with the 3+1 Reissner–Nordström–Kottler solution, in particular existence of asymptotic infinities and horizons. Such a solution was first discovered and studied in [1] (C^3G) [below we refer to formulae in this paper as, for example, to $(1C^3G)$].

Here we find that there exists a simple generalization of the static C^3G solution to the case of a rotating central mass possessing similar geometric

*Departamento de Física, CUCEI, Universidad de Guadalajara, Guadalajara, Jalisco, México; private postal address: Apartado Postal 1-2011, 44100 Guadalajara, Jalisco, México; e-mail: nmitskie@udgserv.cencar.udg.mx

†Departamento de Física, CINVESTAV del IPN, Apartado Postal 14-740, 07000 México, D.F., México; e-mail: aagarcia@fis.cinvestav.mx

and physical properties, as well as being analogous to the Kerr–Newman spacetime with a cosmological term. Since the new solution is a close analogue of the C³G one and its geometric properties are fairly similar to those of the latter, we shall not consider in this paper the horizons and infinities in this spacetime, but concentrate instead on field theoretic problems.

We shall use a 2+1-dimensional spacetime with signature $+$ $-$ $-$ (thus $\det g_{\mu\nu} > 0$) and the definition of the Ricci tensor as $R_{\mu\nu} = R^\alpha{}_{\mu\nu\alpha}$. Then Einstein's equations take the form

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -\kappa T_{\mu\nu} + \Lambda g_{\mu\nu}, \quad (1)$$

while $R = -6\Lambda$ when the trace $T_{\mu\nu}g^{\mu\nu}$ vanishes, hence in our case

$$R_{\mu\nu} = -\kappa T_{\mu\nu} - 2\Lambda g_{\mu\nu}. \quad (2)$$

For a 1-form field A with the skew-symmetric field tensor $F = dA$ this means that the Lagrangian density for this field is $\mathfrak{L} = \sqrt{g}L$, $L = \sigma(-I)^{3/4}$ where $I = F_{\mu\nu}F^{\mu\nu}/2$ is the only first-order differential 1-form field invariant in 2+1. In the case under consideration (the field produced by a rotating charge), $I < 0$.

Taking now the Kerr metric in the Boyer–Lindquist coordinates and fixing the polar angle $\vartheta = \pi/2$, we obtain

$$ds^2 = \frac{\Delta - a^2}{r^2} \left(dt + a \frac{r^2 + a^2 - \Delta}{\Delta - a^2} d\phi \right)^2 - \frac{r^2}{\Delta} dr^2 - \frac{r^2 \Delta}{\Delta - a^2} d\phi^2, \quad (3)$$

$\sqrt{g} = r$. When $a = 0$, this squared interval takes the C³G form for $\Delta_{\text{C}^3\text{G}} = -Mr^2 + \Lambda r^4 + Q^2 r$, so that we only have to add to it a^2 in order to arrive at a (still hypothetical) rotating generalization:

$$\Delta(r) = a^2 + Q^2 r - Mr^2 + \Lambda r^4. \quad (4)$$

It is important to take into account the fact that the dimensionalities of the constants Q and M are here (as well as in [1]) different from those in the Reissner–Nordström or Kerr–Newman metric: the (topological) mass M is dimensionless, and the (nonlinear) charge Q^2 has the dimension of length.

In the natural orthonormal triad $\theta^{(\mu)}$

$$\theta^{(0)} = \frac{\sqrt{\Delta - a^2}}{r} \left(dt + a \frac{r^2 + a^2 - \Delta}{\Delta - a^2} d\phi \right), \quad \theta^{(1)} = \frac{r}{\sqrt{\Delta}} dr, \quad \theta^{(2)} = \frac{r\sqrt{\Delta}}{\sqrt{\Delta - a^2}}, \quad (5)$$

corresponding to the quadratic form (3), the Ricci tensor has non-zero independent components

$$R_{(0)(0)} = -\frac{Q^2}{2r^3} \left(1 + \frac{3a^2}{\Delta - a^2} \right) - 2\Lambda,$$

$$R_{(1)(1)} = \frac{Q^2}{2r^3} + 2\Lambda,$$

$$R_{(2)(2)} = -\frac{Q^2}{2r^3} \left(2 + \frac{3a^2}{\Delta - a^2} \right) + 2\Lambda,$$

$$R_{(0)(2)} = \frac{3aQ^2\sqrt{\Delta}}{2r^3(\Delta - a^2)}$$

(so that the Ricci scalar is $R = -6\Lambda$), thus we find from Einstein's equations (1)

$$\varkappa T_{(0)(0)} = \frac{Q^2}{2r^3} \left(1 + \frac{3a^2}{\Delta - a^2} \right),$$

$$\varkappa T_{(1)(1)} = -\frac{Q^2}{2r^3},$$

$$\varkappa T_{(2)(2)} = \frac{Q^2}{2r^3} \left(2 + \frac{3a^2}{\Delta - a^2} \right),$$

$$\varkappa T_{(0)(2)} = -\frac{3aQ^2\sqrt{\Delta}}{2r^3(\Delta - a^2)}$$

(for triad components, indices in parentheses are used, the triad metric being 2+1 Minkowskian with the signature $+-$, $g_{(\mu)(\nu)} = g^{(\mu)(\nu)}$). We see that the stress-energy tensor is traceless indeed.

Eigenvalues of the stress-energy tensor (here we take this tensor with one co- and one contravariant indices) are: $\lambda_1 = \lambda_2 = Q^2/(2\varkappa r^3)$ and $\lambda_3 = -Q^2/(\varkappa r^3)$. It is characteristic that these eigenvalues do not contain any information about the mass M and angular momentum a . The corresponding eigenvectors $e^{(\mu)}$ (let us consider them as covectors with the components $e^{(\mu)}_{(\alpha)}$ with respect to the usual basis $\theta^{(\alpha)}$, thus $e^{(\mu)} = e^{(\mu)}_{(\alpha)}\theta^{(\alpha)}$) take in terms of $\theta^{(\mu)}$ the form

$$e^{(0)} = N \left(\sqrt{\Delta}\theta^{(0)} - a\theta^{(2)} \right), \quad e^{(1)} = \theta^{(1)}, \quad e^{(2)} = N \left(-a\theta^{(0)} + \sqrt{\Delta}\theta^{(2)} \right) \quad (6)$$

where the normalization factor is $N = (\Delta - a^2)^{-1/2}$. The triad $e^{(\mu)}$ is, naturally, orthonormal due to the symmetry of the stress-energy tensor. The inverse transformation reads

$$\theta^{(0)} = N \left(\sqrt{\Delta} e^{(0)} + a e^{(2)} \right), \quad \theta^{(1)} = e^{(1)}, \quad \theta^{(2)} = N \left(a e^{(0)} + \sqrt{\Delta} e^{(2)} \right) \quad (7)$$

with the same N . A substitution of the expressions (5), after a simple coordinate change $t \rightarrow t + a\phi$, yields the orthonormal covector basis

$$e^{(0)} = \frac{\sqrt{\Delta}}{r} dt, \quad e^{(1)} = \frac{r}{\sqrt{\Delta}} dr, \quad e^{(2)} = r \left(d\phi - \frac{a}{r^2} dt \right), \quad (8)$$

so that the metric tensor becomes much simpler than it was previously, (3):

$$ds^2 = \frac{\Delta}{r^2} dt^2 - \frac{r^2}{\Delta} dr^2 - r^2 \left(d\phi - \frac{a}{r^2} dt \right)^2. \quad (9)$$

We also see that the vector basis corresponding to (8) is

$$X_{(0)} = \frac{r}{\sqrt{\Delta}} \left(\partial_t + \frac{a}{r^2} \partial_\phi \right), \quad X_{(1)} = \frac{\sqrt{\Delta}}{r} \partial_r, \quad X_{(2)} = \frac{1}{r} \partial_\phi. \quad (10)$$

In the new basis, the stress-energy tensor components take the following form:

$$\begin{aligned} T_{(0)(0)} &= \frac{Q^2}{2\kappa r^3} = \lambda_1, \\ T_{(1)(1)} &= -\frac{Q^2}{2\kappa r^3} = -\lambda_1, \\ T_{(2)(2)} &= \frac{Q^2}{\kappa r^3} = -\lambda_3 \end{aligned}$$

(the non-diagonal components are equal to zero). This makes it clear that the respective orthonormalized eigenvectors of $T_{(\beta)}^{(\alpha)}$ are exactly $e^{(0)}$, $e^{(1)}$ and $e^{(2)}$. Now the vanishing of the trace $T = T_{(\mu)(\nu)} g^{(\mu)(\nu)}$ is quite obvious.

While the fulfilment of Einstein's equations guarantees fulfilment of the field equations of the 1-form field (usually labelled as "electromagnetic") since the exact gravitational field theory is automatically self-consistent, it would still be worth considering these 1-form field equations to the end of better understanding of this field's Lagrangian structure. The first (and naïve) idea

coming to the mind is to simply use the Lagrangian density $\mathfrak{L} = \sqrt{g}L(I)$ where $I = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, $F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = dA$, $A = A_\mu dx^\mu$ being the 1-form field potential. But in a 2+1-dimensional spacetime there exists an opportunity to introduce the 1-form f equivalent to 2-form F as

$$\left\{ f_\lambda = \frac{1}{2}E_{\lambda\mu\nu}F^{\mu\nu}, F^{\mu\nu} = E^{\lambda\mu\nu}f_\lambda \right\} \Leftrightarrow \{ f = *F, F = *f \}, \quad (11)$$

the Levi-Civita axial tensor being

$$E_{\lambda\mu\nu} = \sqrt{g}\epsilon_{\lambda\mu\nu}, \quad E^{\lambda\mu\nu} = \frac{1}{\sqrt{g}}\epsilon^{\lambda\mu\nu}, \quad (12)$$

while

$$\epsilon_{\lambda\mu\nu} = \epsilon_{[\lambda\mu\nu]}, \quad \epsilon_{012} = +1 \quad (13)$$

is the Levi-Civita constant three-dimensional symbol (the antisymmetrization in a group of indices is denoted, as usual, with the Bach brackets on the both sides of them); $*$ is the 2+1 Hodge star ($** = +1$).

In order to incorporate into our description the rotation property of the new solution, let us write the field equations as

$$\left(\sqrt{g} \frac{dL}{dI} F^{\mu\nu} \right)_{,\nu} = -4\pi \sqrt{g} j^\mu \quad (14)$$

which are equivalent to

$$d \left(\frac{dL}{dI} f \right) = -4\pi * j. \quad (15)$$

It is known from [1] that the 2+1 nonrotating intrinsically relativistic 1-form field is described by the Lagrangian $L = \sigma(-I)^{3/4}$, ($6C^3G$), where $\sigma = \text{const}$. In this case, the “source term” containing j will be absent, and the equations (14) and (15) become “homogeneous” (in fact, as this will be seen below, the “inhomogeneity” — when it is present — consists of two terms one of which is proportional to A and another, to F , so that its characterization as an inhomogeneity is not completely adequate). It is clear that $j = 0$ manifests the absence of rotation of the 1-form f congruence. Thus, when $a = 0$, the squared interval (9) reduces to that found in [1], and the equations (14), to ($4C^3G$), without the “source” term. However, as we shall see, the

presence of rotation changes the situation drastically. In particular, this change cannot be cast into the Lagrangian form without involvement of a field whose potential is a 2-form (in 2+1).

From the general expression

$$T_\alpha^\beta = \left(2I \frac{dL}{dI} - L \right) \delta_\alpha^\beta - 2 \frac{dL}{dI} f_\alpha f^\beta \quad (16)$$

it can be seen that f is the eigenvector corresponding to the single eigenvalue λ_3 . Thus f has to be proportional to $e^{(2)}$ which rotates, as it follows from (8),

$$f = \frac{Q^{4/3}}{(\kappa\sigma)^{2/3}r^2} e^{(2)}, \quad (17)$$

$f \wedge df \neq 0$, while

$$F = dA = \frac{Q^{4/3}}{(\kappa\sigma)^{2/3}r^2} e^{(0)} \wedge e^{(1)} = \frac{Q^{4/3}}{(\kappa\sigma)^{2/3}r^2} dt \wedge dr, \quad (18)$$

$$A = \frac{Q^{4/3}}{(\kappa\sigma)^{2/3}\sqrt{\Delta}} e^{(0)} = \frac{Q^{4/3}}{(\kappa\sigma)^{2/3}r} dt, \quad (19)$$

$$I = f_\mu f^\mu = -\frac{Q^{8/3}}{(\kappa\sigma)^{4/3}r^4}, \quad \frac{dL}{dI} = -\frac{3\sigma(\kappa\sigma)^{1/3}r}{4Q^{2/3}}. \quad (20)$$

Thus the expression being differentiated in (15) does not represent an exact or closed 1-form due to the rotation parameter a different from zero, as this was to be expected:

$$\frac{dL}{dI} f = -\frac{3(\sigma Q)^{2/3}}{4\kappa^{1/3}} \left(d\phi - \frac{a}{r^2} dt \right) = -\frac{3(\sigma Q)^{2/3}}{4\kappa^{1/3}r} e^{(2)}. \quad (21)$$

It is now obvious that when a rotating field is to be considered, the “homogeneous” variants of field equations (14) and (15) have to be revised (this fact makes it clear why the approach to rotating black holes in [2] is wrong, thus supporting the criticisms given in [3]). This can be done if an additional term would be incorporated into the Lagrangian. The situation is essentially the same as that discovered for a general 3+1 field theoretic description of perfect fluids in [4, 5]. The new term has to leave the stress-energy tensor unchanged, thus (*cf.* [4] with a necessary modification for the

present dimensionality of spacetime) we choose this term to be proportional to the (invariant) combination of completely skew rank-three tensors. Their dependence on the already used objects does not matter (in the spirit of the Noether theorem).

Thus we choose two such tensors: the antisymmetrized product $A_{[\lambda}F_{\mu\nu]}$ [in fact, vanishing due to (18) and (19), but giving nontrivial contributions when differentiated with respect to A and F], and the 2-form field's field tensor (3-form),

$$W = dC, \quad W_{\lambda\mu\nu} = \tilde{W}E_{\lambda\mu\nu} \quad (22)$$

(the Machian or cosmological field, *cf.* for 3+1 [4]). These two objects form the invariant

$$J = W^{\lambda\mu\nu}A_{\lambda}F_{\mu\nu} \equiv 2\tilde{W}A_{\lambda}f^{\lambda} \quad (23)$$

(being equal to zero for the solution under consideration), which we use to construct the additional term M in the Lagrangian: $\mathfrak{L} = \sqrt{g}(L(I) + M(J))$. The stress-energy tensor corresponding to M is

$$T_{\alpha}^{\beta} = \left(2J\frac{dM}{dJ} - M\right)\delta_{\alpha}^{\beta}; \quad (24)$$

the new 1-form field equation takes the form

$$d\left(\frac{dL}{dI}f + \frac{dM}{dJ}\tilde{W}A\right) = -\frac{dM}{dJ}\tilde{W}F, \quad (25)$$

and the (formally) 2-form field equation,

$$d\left(\frac{dM}{dJ}A_{\lambda}f^{\lambda}\right) = 0. \quad (26)$$

The only case when the last equation is fulfilled identically while the equation (25) provides the possibility of rotation, is realised when $M = J$. Then M vanishes due to the properties of our solution, so that the 2-form field's stress-energy tensor (24) vanishes too (the ghost property with respect to Einstein's equations, but not to the Maxwell-type ones); from (25) it follows that

$$d\left(\frac{dL}{dI}f\right) + d\tilde{W} \wedge A + 2\tilde{W}F = 0, \quad (27)$$

yielding

$$\tilde{W} = Sr^2 + \frac{aH}{2r}, \quad S = (\text{arbitrary}) \text{ const.}, \quad H = \sigma \left(\frac{\varkappa\sigma}{Q^2} \right)^{1/3}. \quad (28)$$

This completes the explicit proof of self-consistency of our solution, although the fact of fulfilment of Einstein's equations already could not leave any room for doubts.

Since it is more common to use the field equation in the form (14), with a non-zero right-hand side in the rotating case, we now calculate

$$\begin{aligned} \sqrt{g} \frac{dL}{dI} F^{\mu\nu} &= \frac{3(\sigma Q)^{2/3}}{4\varkappa^{1/3}} \left(\epsilon^{\phi\mu\nu} - \frac{a}{r^2} \epsilon^{t\mu\nu} \right) \\ &\equiv -\frac{3a(\sigma Q)^{2/3}}{4\varkappa^{1/3}r^2} \left(\delta_r^\mu \delta_\phi^\nu - \delta_\phi^\mu \delta_r^\nu \right) + \text{const}^{\mu\nu}. \end{aligned}$$

Thus

$$j = \frac{3a(\sigma Q)^{2/3}}{8\pi\varkappa^{1/3}r^3} X_{(2)} = \frac{3a(\sigma Q)^{2/3}}{8\pi\varkappa^{1/3}r^4} \partial_\phi. \quad (29)$$

We see that the source term, usually interpreted as the electric current density, is spacelike (hence not attributable to any moving charges). It is proportional to the rotation parameter a , though it also contains the coefficient $Q^{2/3}$, but the latter is related to the central pointlike charge and by no means to any spatial distribution of charges and/or currents. We interpret the role of Q as that of a “catalyst” of rotation which otherwise (in the absence of an “electric” field) cannot manifest itself in the (then absent) Maxwell-type equations. Moreover, the “source” term in (27) [or, equivalently, in (15)] does linearly depend on the 1-form field potential and, in its other summand, on the corresponding field tensor, — the fact being completely foreign to the basic ideas of the conventional electromagnetic field theory (and there is no other — alternative — solution to this problem).

The exact solution of a field generated by an obviously pointlike source in 2+1-dimensional spacetime, when a rotation is present, belongs to a distributed-source containing system of Maxwell-type equations. This fact leads to a radical revision of the physical interpretation of distributed sources of 1-form fields in 2+1 (*not as electric current densities*), as well as to a revision of the physical meaning of these very fields (there is essentially less analogy between a 1-form field in 2+1 and the electromagnetic one in 3+1

than it is usually admitted. In particular, the “magnetic” type fields in fact describe here 2+1 perfect fluids: *cf.* a statement made in [4, 5] which should be however somewhat modified in the “electric” case just having been considered). The system of fields describing a rotating massive charged centre thus consists of the gravitational field (g), the 1-form field (A), and the 2-form field [C , being a ghost field in the sense of Einstein’s equations, but inevitable to fulfil the Maxwell-type equations (25) to incorporate the rotation]. A more general and complete treatment of the 1-form field theory in 2+1 will be found in a subsequent publication.

Acknowledgements

A.A.G. acknowledges a partial support from the CONACyT (Mexico) Project 32138E; N.V.M. thanks CINVESTAV IPN for hospitality during his visit there when this work was done.

References

- [1] M. Cataldo, N. Cruz, S. del Campo, A. Garcia (2000) *Phys. Lett.* **B 484**, 154.
- [2] M. Bañados, C. Teitelboim and J. Zanelli (1992) *Phys. Rev. Lett.* **69**, 1849; M. Bañados, M. Henneaux, C. Teitelboim and J. Zanelli (1993) *Phys. Rev.* **D 48**, 1506; C. Martínez, C. Teitelboim and J. Zanelli (1999) *Charged Rotating Black hole in Three Spacetime Dimensions*, hep-th/9912259.
- [3] A. García (1999) *On the Rotating Charged BTZ Metric*, hep-th/9909111.
- [4] N. V. Mitskievich (1999) *Int. J. Theor. Phys.* **38**, 997; N. V. Mitskievich (1998) gr-qc/9811077.
- [5] N. V. Mitskievich (1999) *Gen. Rel. Grav.* **31**, 713.